Homotopy Analysis Method for Ablowitz–Ladik Lattice

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In this paper, the homotopy analysis method is successfully applied to solve the systems of differential-difference equations. The Ablowitz–Ladik lattice system are chosen to illustrate the method. Comparisons between the results of the proposed method and exact solutions reveal that the homotopy analysis method is very effective and simple in solving systems of differential-difference equations.

Key words: Homotopy Analysis Method (HAM); Ablowitz–Ladik Lattice System.

1. Introduction

It is well known that the investigation of differential-difference equations (DDEs) which describe many important phenomena and dynamical processes in many different fields, such as particle vibrations in lattices, currents in electrical networks, pulses in biological chains, and so on, has played an important role in the study of modern physics. Unlike difference equations which are fully discretized, DDEs are semi-discretized with some (or all) of their spacial variables discretized while time is usually kept continuous and then also play an important role in numerical simulations of nonlinear partial differential equations, queuing problems, and discretization in solid state and quantum physics. There is a vast body of work on DDEs [1 – 12].

For better understanding the meaning of DDEs, it is crucial to search for exact analytic solutions of DDEs. Since the work of Wadati in the 1970s [2], many powerful methods have been generalized to construct solutions of DDEs such as Bäcklund transformation [13 – 15], Darboux transformation [16], Hirota method [17], etc.

In 1992, based on the idea of homotopy in topology, Liao [18] proposed a general analytic method for nonlinear problems, namely the homotopy analysis method (HAM). Unlike the traditional methods (for example, perturbation techniques and so on), the HAM contains many auxiliary parameters which provide us with a simple way to adjust and control the convergence region and rate of convergence of the series solution and has been successfully employed to solve explicit analytic solutions for many types of nonlinear problems [19 – 24].

Motivated by the publications above, we would like to extend the applications of the HAM to systems of differential-difference equations. For illustration, we apply it to Ablowitz–Ladik lattice system which is the discretization of the nonlinear Schrödinger equation and can be solved by the Bäcklund and Darboux transformation [25, 26].

This paper is organized as follows: In Section 2, a brief outline of the generalized HAM for a system of DDEs with initial condition is presented. In Section 3, we apply the proposed method to the Ablowitz–Ladik lattice system to verify the effectiveness of it and also give the proof of convergence theorem. In Section 4, a brief analysis of the obtained results is given. A short summary and discussion are presented in final.

2. HAM for a System of DDEs

For illustration, we consider the following system of DDEs:

$$\mathcal{N}_i[u_{i,n}(t), u_{i,n-1}(t), u_{i,n+1}(t), \ldots] = 0\,, \qquad (1)$$

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where $\mathcal{N}_i$ are nonlinear differential operators that represent the whole system of equations, $n \in N$ and t denote independent variables, and $u_{i,n}(t)$ are unknown functions, respectively. By means of HAM, we construct the so-called zero-order deformation equations

$$(1-q)\mathcal{L}_i[\phi_{i,n}(t;q)-u_{i,n,0}(t)] = qhH_{i,n}(t) \cdot \mathcal{N}_i[\phi_{i,n}(t;q),\phi_{i,n-1}(t;q),\phi_{i,n+1}(t;q),\ldots], \tag{2}$$

where $q \in [0,1]$ is an embedding parameter, h is a nonzero auxiliary parameter, $H_{i,n}(t)$ are nonzero auxiliary functions, $\mathcal{L}_i$ are auxiliary linear operators, $u_{i,n,0}(t)$ are initial guesses of $u_{i,n}(t)$, $\phi_{i,n}(t;q)$ are unknown functions on independent variables n, t, and q. It is important to note that one has great freedom to choose auxiliary parameters such as h in HAM. Obviously, when the embedding parameter q increases from 0 to 1, $\phi_{i,n}(t;q)$ vary (or deforms) continuously from the initial guesses $\phi_{i,n}(t;0) = u_{i,n,0}(t)$ to the exact solutions $\phi_{i,n}(t;1) = u_{i,n}(t)$ of the original system (1).

Define the so-called mth-order deformation derivatives

$$u_{i,n,m}(t) = \frac{1}{m!}\frac{\partial^m \phi_{i,n}(t;q)}{\partial q^m}\mid_{q=0}. \tag{3}$$

Expanding $\phi_{i,n}(t;q)$ in Taylor series with respect to the embedding parameter q, we have

$$\phi_{i,n}(t;q) = u_{i,n,0}(t) + \sum_{m=1}^{\infty} u_{i,n,m}(t)q^m. \tag{4}$$

Then, correspondingly

$$\phi_{i,n-k}(t;q) = u_{i,n-k,0}(t) + \sum_{m=1}^{\infty} u_{i,n-k,m}(t)q^m, \tag{5}$$

$$\phi_{i,n+k}(t;q) = u_{i,n+k,0}(t) + \sum_{m=1}^{\infty} u_{i,n+k,m}(t)q^m, \quad k \in N. \tag{6}$$

If the auxiliary linear operator, the initial guesses, the auxiliary parameter h, and the auxiliary functions $H_{i,n}(t)$ are properly chosen, the Series (4) converge at $q = 1$, one has

$$u_{i,n}(t) = u_{i,n,0}(t) + \sum_{m=1}^{\infty} u_{i,n,m}(t), \tag{7a}$$

$$u_{i,n-k}(t) = u_{i,n-k,0}(t) + \sum_{m=1}^{\infty} u_{i,n-k,m}(t), \tag{7b}$$

$$u_{i,n+k}(t) = u_{i,n+k,0}(t) + \sum_{m=1}^{\infty} u_{i,n+k,m}(t), \tag{7c}$$

which must be one of the solutions of the original nonlinear equations, as proved by Liao [22]. As $hH_{i,n}(t) = -1$, (2) becomes

$$(1-q)\mathcal{L}_i[\phi_{i,n}(t;q)-u_{i,n,0}(t)] + q\mathcal{N}_i[\phi_{i,n}(t;q),\phi_{i,n+1}(t;q),\phi_{i,n-1}(t;q),\ldots] = 0, \tag{8}$$

which is mostly used in the homotopy-perturbation method [27].

For brevity, define the vectors

$$\vec{u}_{i,n,m}(t) = \{u_{i,n,0}(t),u_{i,n,1}(t),\ldots,u_{i,n,m}(t)\}, \tag{9a}$$

$$\vec{u}_{i,n-k,m}(t) = \{u_{i,n-k,0}(t),u_{i,n-k,1}(t),\ldots,u_{i,n-k,m}(t)\}, \tag{9b}$$

$$\vec{u}_{i,n+k,m}(t) = \{u_{i,n+k,0}(t),u_{i,n+k,1}(t),\ldots,u_{i,n+k,m}(t)\}. \tag{9c}$$

Differentiating the zero-order deformation (2) m times with respect to q and then dividing them by $m!$ and finally setting $q = 0$, we have the mth-order deformation equations

$$\mathcal{L}_i[u_{i,n,m}(t) - \chi_{n,m}u_{i,n,m-1}(t)] = hH_{i,n}(t) \cdot R_{i,m}(\vec{u}_{i,n,m-1}(t),\vec{u}_{i,n-1,m-1}(t),\vec{u}_{i,n+1,m-1}(t),\ldots), \tag{10}$$

where

$$R_{i,m}(\vec{u}_{i,n,m-1}(t),\vec{u}_{i,n-1,m-1}(t),\vec{u}_{i,n+1,m-1}(t),\ldots) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}_i[\phi_{i,n}(t;q)]}{\partial q^{m-1}}\mid_{q=0} \tag{11}$$

and

$$\chi_{n,m} = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{12}$$

It should be emphasized that $u_{i,n,m}(t)$ $(m \ge 1)$ is governed by the linear Equation (10) with the linear boundary conditions that come from the original problem. Thus we can obtain $u_{i,n,1}(t)$, $u_{i,n,2}(t),\ldots$ by solving the linear high-order deformation (10) one after the other in order, based on symbolic computation softwares such as Maple, Mathematica, and so on.

3. Application to the Ablowitz–Ladik Lattice System

In this section, to verify the validity and the effectiveness of HAM in solving system of DDEs, we apply it to the Ablowitz–Ladik lattice system

$$\frac{\partial u_n}{\partial t} = (\alpha + u_n v_n)(u_{n+1} + u_{n-1}) - 2\alpha u_n\,, \tag{13a}$$

$$\frac{\partial v_n}{\partial t} = -(\alpha + u_n v_n)(v_{n+1} + v_{n-1}) + 2\alpha v_n\,, \tag{13b}$$

subject to the initial conditions

$$u_n(0) = \frac{\alpha \sinh^2(d)}{\beta}(1 - \tanh(dn + \delta))\,, \tag{14a}$$

$$v_n(0) = \beta(1 + \tanh(dn + \delta))\,, \tag{14b}$$

whose exact solutions can be written as [28]

$$u_n(t) = \frac{\alpha \sinh^2(d)}{\beta} \cdot (1 - \tanh(dn - 2\alpha \sinh^2(d)t + \delta))\,, \tag{15a}$$

$$v_n(t) = \beta(1 + \tanh(dn - 2\alpha \sinh^2(d)t + \delta))\,. \tag{15b}$$

Here, $u_n(t)$ and $v_n(t)$ are functions of continuous time variable t and discrete variable n.

To solve System (13)–(14) by means of HAM, we choose the initial guesses

$$u_{n,0}(t) = \frac{\alpha \sinh^2(d)}{\beta}(1 - \tanh(dn + \delta))\,, \tag{16a}$$

$$v_{n,0}(t) = \beta(1 + \tanh(dn + \delta))\,, \tag{16b}$$

and the auxiliary linear operator

$$\mathcal{L}[\phi_{i,n}(t;q)] = \frac{\partial \phi_{i,n}(t;q)}{\partial t}, \quad i = 1, 2\,, \tag{17}$$

with the property

$$\mathcal{L}[c_i] = 0\,, \tag{18}$$

where c_i $(i = 1, 2)$ are integral constants. Furthermore, System (13) suggests that we define a system of nonlinear operators as

$$\mathcal{N}_1[\phi_{i,n}(t;q), \phi_{i,n-1}(t;q), \phi_{i,n+1}(t;q), \ldots] = \frac{\partial \phi_{1,n}(t;q)}{\partial t} - (\alpha + \phi_{1,n}(t;q)\phi_{2,n}(t;q)) \cdot (\phi_{1,n+1}(t;q) + \phi_{1,n-1}(t;q)) + 2\alpha \phi_{1,n}(t;q)\,, \tag{19a}$$

$$\mathcal{N}_2[\phi_{i,n}(t;q), \phi_{i,n-1}(t;q), \phi_{i,n+1}(t;q), \ldots] = \frac{\partial \phi_{2,n}(t;q)}{\partial t} + (\alpha + \phi_{1,n}(t;q)\phi_{2,n}(t;q)) \cdot (\phi_{2,n+1}(t;q) + \phi_{2,n-1}(t;q)) - 2\alpha \phi_{2,n}(t;q)\,. \tag{19b}$$

Using above definitions, we construct the zeroth-order deformation equations

$$(1-q)\mathcal{L}[\phi_{1,n}(t;q) - u_{n,0}(t)] = qhH_{1,n}(t) \cdot \mathcal{N}_1[\phi_{i,n}(t;q), \phi_{i,n-1}(t;q), \phi_{i,n+1}(t;q), \ldots]\,, \tag{20a}$$

$$(1-q)\mathcal{L}[\phi_{2,n}(t;q) - v_{n,0}(t)] = qhH_{2,n}(t) \cdot \mathcal{N}_2[\phi_{i,n}(t;q), \phi_{i,n-1}(t;q), \phi_{i,n+1}(t;q), \ldots]\,, \tag{20b}$$

with the initial conditions

$$\phi_{1,n}(0;q) = u_{n,0}(0), \quad \phi_{2,n}(0;q) = v_{n,0}(0)\,, \tag{21}$$

where $q \in [0,1]$ denotes an embedding parameter, $h \neq 0$ is an auxiliary parameter and $H_{i,n}(t)$ are auxiliary functions. Obviously, when $q = 0$ and $q = 1$

$$\phi_{1,n}(t;0) = u_{n,0}(t), \quad \phi_{1,n}(t;1) = u_n(t)\,, \tag{22a}$$

$$\phi_{2,n}(t;0) = v_{n,0}(t), \quad \phi_{2,n}(t;1) = v_n(t)\,. \tag{22b}$$

Therefore, as the embedding parameter q increases continuously from 0 to 1, $\phi_{i,n}(t;q)$ vary from the initial guesses $u_{n,0}(t)$ and $v_{n,0}(t)$ to the solutions $u_n(t)$ and $v_n(t)$. Expanding $\phi_{i,n}(t,q)$ in Taylor series with respect to q one has

$$\phi_{1,n}(t;q) = u_{n,0}(t) + \sum_{m=1}^{\infty} u_{n,m}(t) q^m\,, \tag{23a}$$

$$\phi_{2,n}(t;q) = v_{n,0}(t) + \sum_{m=1}^{\infty} v_{n,m}(t) q^m\,, \tag{23b}$$

where

$$u_{n,m}(t) = \frac{1}{m!} \frac{\partial^m \phi_{1,n}(t;q)}{\partial q^m} \Big|_{q=0}\,, \quad v_{n,m}(t) = \frac{1}{m!} \frac{\partial^m \phi_{2,n}(t;q)}{\partial q^m} \Big|_{q=0}\,. \tag{24}$$

If the auxiliary parameters h and $H_{i,n}(t)$ are properly chosen, above Series (23) are convergent at $q = 1$. Then one has

$$u_n(t) = \sum_{m=0}^{\infty} u_{n,m}(t), \quad v_n(t) = \sum_{m=0}^{\infty} v_{n,m}(t)\,, \tag{25}$$

and we will prove at the end of this section that they must be solutions of the original system.

Now, we define the vectors

$$\vec{u}_{n,m}(t) = \{u_{n,0}(t), u_{n,1}(t), \ldots, u_{n,m}(t)\}\,, \quad \vec{v}_{n,m}(t) = \{v_{n,0}(t), v_{n,1}(t), \ldots, v_{n,m}(t)\}\,. \tag{26}$$

So the mth-order deformation equations are

$$\begin{aligned}&\mathcal{L}[u_{n,m}(t)-\chi_{n,m}u_{n,m-1}(t)]=hH_{1,n}(t)\\&\cdot\mathcal{R}_{1,m}[\vec{u}_{n,m-1}(t),\vec{v}_{n,m-1}(t),\vec{u}_{n-1,m-1}(t),\\&\vec{v}_{n-1,m-1}(t),\vec{u}_{n+1,m-1}(t),\vec{v}_{n+1,m-1}(t),\dots],\end{aligned}\tag{27a}$$

$$\begin{aligned}&\mathcal{L}[v_{n,m}(t)-\chi_{n,m}v_{n,m-1}(t)]=hH_{2,n}(t)\\&\cdot\mathcal{R}_{2,m}[\vec{u}_{n,m-1}(t),\vec{v}_{n,m-1}(t),\vec{u}_{n-1,m-1}(t),\\&\vec{v}_{n-1,m-1}(t),\vec{u}_{n+1,m-1}(t),\vec{v}_{n+1,m-1}(t),\dots],\end{aligned}\tag{27b}$$

with the initial conditions

$$u_{n,m}(0)=0,\quad v_{n,m}(0)=0,\quad m\geq 1,\tag{28}$$

where

$$\begin{aligned}&\mathcal{R}_{1,m}[\vec{u}_{n,m-1}(t),\vec{v}_{n,m-1}(t),\vec{u}_{n-1,m-1}(t),\vec{v}_{n-1,m-1}(t),\\&\vec{u}_{n+1,m-1}(t),\vec{v}_{n+1,m-1}(t),\dots]=\frac{\partial u_{n,m-1}}{\partial t}\\&-\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}u_{n,i}v_{n,j-i}\right)(u_{n+1,m-1-j}+u_{n-1,m-1-j})\\&-\alpha(u_{n+1,m-1}-2u_{n,m-1}+u_{n-1,m-1}),\end{aligned}\tag{29a}$$

$$\begin{aligned}&\mathcal{R}_{2,m}[\vec{u}_{n,m-1}(t),\vec{v}_{n,m-1}(t),\vec{u}_{n-1,m-1}(t),\vec{v}_{n-1,m-1}(t),\\&\vec{u}_{n+1,m-1}(t),\vec{v}_{n+1,m-1}(t),\dots]=\frac{\partial v_{n,m-1}}{\partial t}\\&+\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}u_{n,i}v_{n,j-i}\right)(v_{n+1,m-1-j}+v_{n-1,m-1-j})\\&+\alpha(v_{n+1,m-1}-2v_{n,m-1}+v_{n-1,m-1}),\end{aligned}\tag{29b}$$

and $\chi_{n,m}$ satisfy (12).

In order to obey both the rule of solution expression and the rule of the coefficient ergodicity [22], the corresponding auxiliary functions can be determined uniquely $H_{i,n}(t)=1$.

It should be emphasized that $u_{n,m}(t)$ and $v_{n,m}(t)$ $(m\geq 1)$ are governed by the linear Equation (27) with the linear initial Conditions (28). Thus we can get all $u_{n,m}(t)$ and $v_{n,m}(t)$ $(m\geq 1)$ easily and according to (25), we can get the solutions of Systems (13) and (14).

Then, HAM for the system of DDEs provides us with a family of solution expression in the auxiliary parameter h. The convergence region of solution series depend upon the value of h. Next, we will illustrate the convergence theorem and prove it.

Theorem 3.1 *Convergence Theorem*

The Series (25) are exact solutions of (13) and (14) as long as they are convergent.

Proof. Since $u_n(t)=\sum_{m=0}^{\infty}u_{n,m}(t)$ and $v_n(t)=\sum_{m=0}^{\infty}v_{n,m}(t)$ is convergent, we must have

$$\lim_{m\to\infty}u_{n,m}(t)=0,\quad \lim_{m\to\infty}v_{n,m}(t)=0.\tag{30}$$

Due to Definitions (11) of $\chi_{n,m}$, and the mth-order deformation Equation (27), it holds

$$\begin{aligned}&hH_{1,n}(t)\sum_{m=1}^{\infty}\mathcal{R}_1[\vec{u}_{n,m-1}(t),\vec{v}_{n,m-1}(t),\vec{u}_{n-1,m-1}(t),\\&\vec{v}_{n-1,m-1}(t),\vec{u}_{n+1,m-1}(t),\vec{v}_{n+1,m-1}(t),\dots]\\&=\lim_{m\to\infty}\mathcal{L}[u_{n,m}(t)]=\mathcal{L}\Big[\lim_{m\to\infty}u_{n,m}(t)\Big]=0,\end{aligned}\tag{31}$$

$$\begin{aligned}&hH_{2,n}(t)\sum_{m=1}^{\infty}\mathcal{R}_2[\vec{u}_{n,m-1}(t),\vec{v}_{n,m-1}(t),\vec{u}_{n-1,m-1}(t),\\&\vec{v}_{n-1,m-1}(t),\vec{u}_{n+1,m-1}(t),\vec{v}_{n+1,m-1}(t),\dots]\\&=\lim_{m\to\infty}\mathcal{L}[v_{n,m}(t)]=\mathcal{L}\Big[\lim_{m\to\infty}v_{n,m}(t)\Big]=0,\end{aligned}\tag{32}$$

which give

$$\begin{aligned}&\sum_{m=1}^{\infty}\mathcal{R}_{1,m}[\vec{u}_{n,m-1}(t),\vec{v}_{n,m-1}(t),\vec{u}_{n-1,m-1}(t),\\&\vec{v}_{n-1,m-1}(t),\vec{u}_{n+1,m-1}(t),\vec{v}_{n+1,m-1}(t),\dots]=0,\end{aligned}\tag{33}$$

$$\begin{aligned}&\sum_{m=1}^{\infty}\mathcal{R}_{2,m}[\vec{u}_{n,m-1}(t),\vec{v}_{n,m-1}(t),\vec{u}_{n-1,m-1}(t),\\&\vec{v}_{n-1,m-1}(t),\vec{u}_{n+1,m-1}(t),\vec{v}_{n+1,m-1}(t),\dots]=0,\end{aligned}\tag{34}$$

because both of the auxiliary parameter h and the auxiliary functions $H_{i,n}(t)$ are nonzero. Substituting Definitions (29) of R_i into above expressions, we have

$$\begin{aligned}&\sum_{m=1}^{\infty}\left(\frac{\partial u_{n,m-1}}{\partial t}-\alpha(u_{n+1,m-1}-2u_{n,m-1}+u_{n-1,m-1})\right)\\&-\sum_{m=1}^{\infty}\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}u_{n,i}v_{n,j-i}\right)(u_{n+1,m-1-j}+u_{n-1,m-1-j})\\&=\frac{\partial}{\partial t}\sum_{m=1}^{\infty}u_{n,m-1}-\alpha\sum_{m=1}^{\infty}(u_{n+1,m-1}-2u_{n,m-1}+u_{n-1,m-1})\\&-\sum_{m=1}^{\infty}\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j}u_{n,i}v_{n,j-i}\right)(u_{n+1,m-1-j}+u_{n-1,m-1-j})\\&=\frac{\partial}{\partial t}\sum_{m=0}^{\infty}u_{n,m}-\alpha\sum_{m=0}^{\infty}(u_{n+1,m}-2u_{n,m}+u_{n-1,m})\\&-\sum_{m=0}^{\infty}\sum_{j=0}^{m}\left(\sum_{i=0}^{j}u_{n,i}v_{n,j-i}\right)(u_{n+1,m-j}+u_{n-1,m-j})\end{aligned}\tag{35}$$

$$= \frac{\partial}{\partial t}\sum_{m=0}^{\infty} u_{n,m} - \alpha \sum_{m=0}^{\infty}(u_{n+1,m} - 2u_{n,m} + u_{n-1,m}) - \sum_{m=0}^{\infty} u_{n,m} \sum_{m=0}^{\infty} v_{n,m}\left(\sum_{m=0}^{\infty} u_{n+1,m} + \sum_{m=0}^{\infty} u_{n-1,m}\right)$$

and also

$$\sum_{m=1}^{\infty}\left(\frac{\partial v_{n,m-1}}{\partial t} + \alpha(v_{n+1,m-1} - 2v_{n,m-1} + v_{n-1,m-1})\right) + \sum_{m=1}^{\infty}\sum_{j=0}^{m-1}\left(\sum_{i=0}^{j} u_{n,i}v_{n,j-i}\right)(v_{n+1,m-1-j} + v_{n-1,m-1-j}) \quad (36)$$
$$= \frac{\partial}{\partial t}\sum_{m=0}^{\infty} v_{n,m} + \alpha \sum_{m=0}^{\infty}(v_{n+1,m} - 2v_{n,m} + v_{n-1,m}) - \sum_{m=0}^{\infty} u_{n,m} \sum_{m=0}^{\infty} v_{n,m}\left(\sum_{m=0}^{\infty} v_{n+1,m} + \sum_{m=0}^{\infty} v_{n-1,m}\right),$$

which means $\sum_{m=0}^{\infty} u_{n,m}$ and $\sum_{m=0}^{\infty} u_{v,m}$ admit the System (13). Besides, using the initial Conditions (28) and the Definitions (14) of the initial guesses, we have

$$\sum_{m=0}^{\infty} u_{n,m}(0) = u_{n,0}(0) = u_n(0) = \frac{\alpha \sinh^2(d)}{\beta}(1 - \tanh(dn+\delta)), \quad (37)$$

$$\sum_{m=0}^{\infty} v_{n,m}(0) = v_{n,0}(0) = v_n(0) = \beta(1 + \tanh(dn+\delta)).$$

Thus, due to (35)–(37), the series $\sum_{m=0}^{\infty} u_{n,m}(t)$ and $\sum_{m=0}^{\infty} v_{n,m}(t)$ must be exact solutions of Systems (13) and (14). This ends the proof.

4. Results Analysis

It has been proved that, as long as a series solution given by HAM converges, it must be one of the exact solutions [22]. So the validity of HAM is based on such an assumption that the Series (4) converge at $q = 1$ which can be ensured by the properly chosen auxiliary parameter h. In general, by means of the so-called h-curve [22], it is straightforward to choose a proper value of h.

In Figure 1, we plot the h-curve for 6th-order HAM approximations of (25) at $\alpha = \beta = \delta = 1$, $d = 0.1$, $n = 10$, and $t = 0.1$. By HAM, it is easy to discover the valid region of h, which corresponds to the line segments nearly parallel to the horizontal axis. From this figure, we could find that if h is about in area $[-2, 0.6]$ the result is convergent.

To increase the accuracy and convergence of the solution, Liao [20] has developed a new technique,

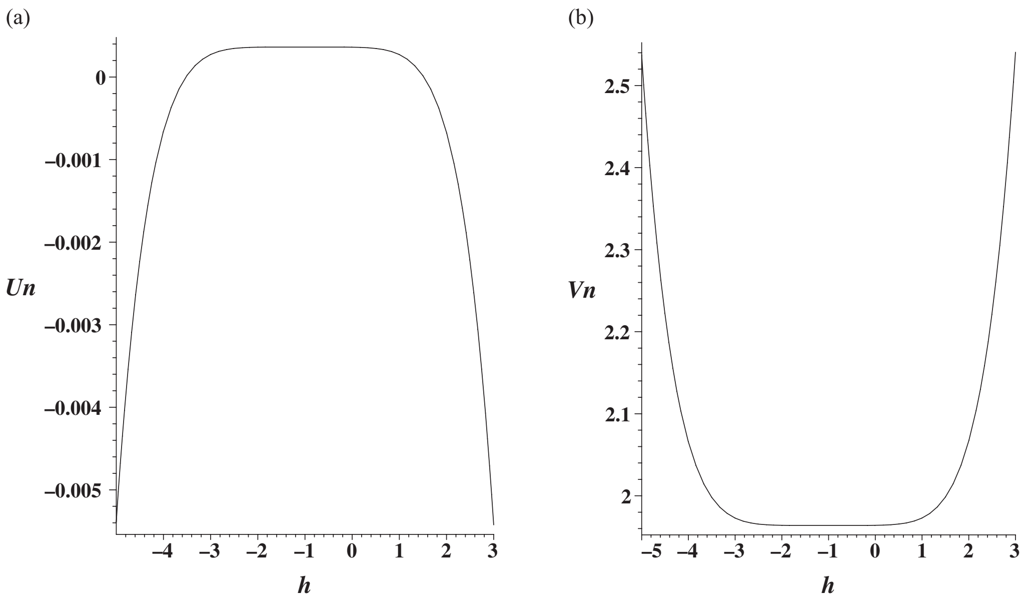

Fig. 1. h-curve for 6th-order HAM approximations of (25): (a) h-curve for 6th-order HAM approximation $u_n(t)$ and (b) h-curve for 6th-order HAM approximation $v_n(t)$, when $\alpha = \beta = \delta = 1$, $d = 0.1$, $n = 10$, and $t = 0.1$.

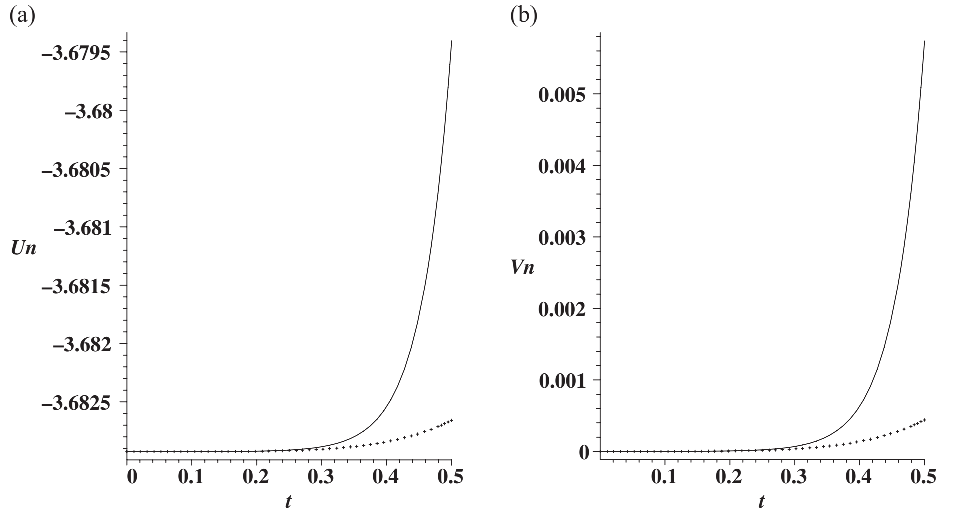

Fig. 2. Comparisons of the exact solutions with the HP approximations: (a) between the exact solution $u_n(t)$ and $[3,3]$ HP approximations; (b) between the exact solution $v_n(t)$ and $[3,3]$ HP approximations, when $h=-1.2$, $\alpha=-4$, $\beta=3$, $d=-1$, and $\delta=n=1$. Dotted line: $[3,3]$ HP approximations; solid line: exact solutions.

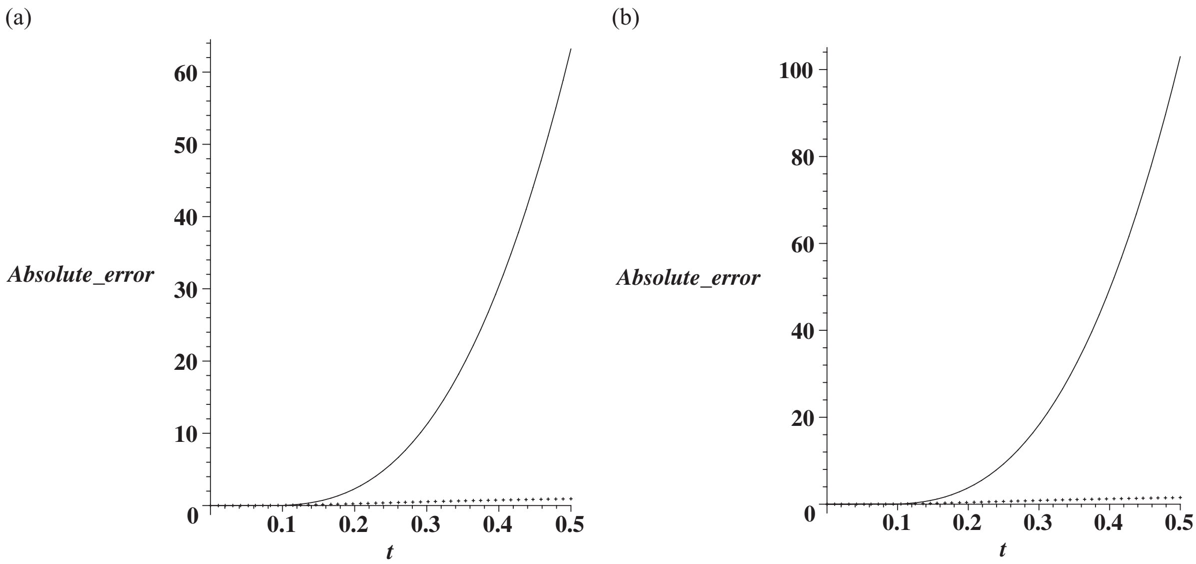

Fig. 3. Comparisons between absolute errors of $[2,2]$ HP and 4th-order HAM approximations: (a) for solution $u_n(t)$; (b) for solution $v_n(t)$, when $h=-1.2$, $\alpha=-4$, $\beta=3$, $d=-1$, and $\delta=n=1$. Dotted line: absolute error of $[2,2]$ HP approximations; solid line: absolute error of 4th-order HAM approximations.

namely the homotopy-Padé (HP) method. Here comparisons are made between the $[3,3]$ HP approximations and exact solutions, when $h=-1.2$, $\alpha=-4$, $\beta=3$, $d=-1$, and $\delta=n=1$, as shown in Figure 2. From this figure, the approximations obtained by the HP method agree well with the exact solutions when t tends to $t=0$.

In Figure 3, to verify the effectiveness of the HP method, comparisons are made between absolute errors of the $[2,2]$ HP and 4th-order HAM approximations. It can easily draw a conclusion that HP method is an effective method to accelerate the convergence of the result and enlarge the convergence field.

5. Conclusion

In this paper, we successfully generalize the HAM to solve a system of DDEs. For illustration, the proposed method is applied to solve the Ablowitz–Ladik lattice system. Numerical results show that the HAM provides a very effective method and a promising tool for solving a system of DDEs. The advantage of HAM is high flexibility in choosing the auxiliary parameter which provides a convenient way for controlling the convergence region of the series solutions. The power series has often finite radius of convergence. So, one must apply the HP technique to enlarge the convergence-region. Actually it would be much better to use exponential functions as base functions. And we will try other auxiliary linear operators in following works.

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